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SOME PROBLEMS IN THE ALLOCATION OF INTERCEPTORS IN THE DEFENSE OF A TASK GROUP

by

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1. Introduction

We consider here the defense of a carrier task group against enemy air attacks that are large and all of the same type. The word "large" will be defined below; the assumption of largeness makes possible certain simplifications. By "all of the same type" is meant that the problem of selecting weapons for use against different types of aircraft is not considered here.

The first part of the paper (sections 2-7) deals with the problem of integrating the first two phases (interceptors and guided missiles) of the defense. It is brought out here that the two phases are related through the fact that an interceptor has at least the following two objectives:

(i) to kill a bomber (To "kill" will mean to prevent the bomber from delivering its bombs.)

(ii) to break up the enemy formation so as to make it a better target for guided missiles and anti-aircraft.

It is conceivable that in certain circumstances objective (ii) may be impossible; when this is the case the interceptor and guided-missile phases become independent and the entire problem is simplified. Three different models are used to provide a simple mathematical expression for the effect of (ii). In terms of these models we answer the question, "Should the enemy always bring in all of his aircraft in one raid (to be called a "maximum raid") or should he sometimes divide his forces so as to persuade the defense to adopt a policy of saving some of its forces for a later attack that may not come?" In answering this question we automatically deal with the problem of allocation of Combat Air Patrol.

Since the first part lends support to the natural view that the maximum raid is in many cases most effective, the second part (sections 8-14) is devoted to the case of one large raid. Here it is assumed that the enemy will attempt to have all of his M available aircraft arrive at some time μ (unknown to the defense), but that it will be difficult for him to achieve this simultaneity, and there will be a dispersion σ in the arrival times of attacking bombers. Detection times of the earliest arrivals enable the defense to form estimates of $M, \mu,$ and σ, but it is shown that the defense cannot obtain useful estimates of those parameters unless he knows something about at least one of them from operational experience or from intelligence. Unfortunately, statistical estimation here demands a known (preferably approximately normal) type of distribution of detection times, but the discussion nevertheless brings out the kind of information that is needed during the engagement and describes a method of displaying it.

2. Notation

We let

M = total number of enemy aircraft participating in one raid or in two raids close together in time.

N = total number of airborne CAP when first raid is detected.

p_1 = probability that an interceptor will kill an attacking aircraft. (It will be assumed that speeds are such that no interceptor will kill two aircraft in one mission.)

m = number of guided missile salvos that can be fired against one raid which consists of aircraft arriving almost simultaneously. (This number is only a first approximation to realism; in practice it may vary with several factors such as the direction or directions from which enemy aircraft approach.)

3. <u>The Problem of One Raid vs. Two Raids</u>

The advantages of the maximum raid are those of surprise, saturation of defense systems, and minimum CAP interference in case the defense decides to save some CAP for possible later raids. In order to obtain the latter advantage, the enemy must occasionally send more than one raid. If R_1 and R_2 denote the first two raids, R_2 must not be detected too soon after R_1, else they will constitute essentially one raid - e.g., if all N CAP were initially sent against R_1 it would not be too late to divert some to R_2. On the other hand, R_2 must not arrive too long after detection of R_1, as additional interceptors alerted by detection of R_1 may then be airborne. The enemy is faced with a difficult timing problem which is complicated by the fact that detection times are not entirely at his disposal. For this reason, as well as for mathematical simplicity, we shall suppose that the enemy chooses only between sending a maximum raid of size M or two raids of size $M/2$ each. Results will cover somewhat more general situations, since in cases where two equal raids should never be used, the same will probably be true of two unequal raids or of three or more raids.

4. Models Based on "Probability of Break-up"

Two models will be considered here for describing the effect of interceptors in breaking up an enemy formation in such a way that the formation becomes a better target for guided missiles. Since this effect is a complicated one, the simple models used below to describe it are to be construed as merely helpful rather than accurate.

Model A. We assume here that the break-up is with respect to time, so that a formation that has been broken up will straggle in, relatively speaking. If the formation is unbroken there will be time for only m guided missile salvos against it, while if the formation is broken there will be time for one salvo against each aircraft. Intermediate possibilities will be ignored. We let

$$p_2 = \text{probability of kill for a guided missile salvo.}$$

Model B. In this case the break-up is considered to be with respect to distance. It is assumed that the formation of bombers is tight enough so that guided missile radars may not be able to resolve individual aircraft. The interceptors cannot slow down the formation but may be able to break it up in the sense of causing it to spread out enough to allow resolution. Again we ignore intermediate cases. We let

p_3 = probability, when formation is unbroken, that a missile salvo will kill a bomber

p_4 = probability, when formation is broken, that a missile salvo will kill a bomber

and we suppose in either case that there is time for only m salvos.

In order to simplify simultaneous treatment of these two models, we let

$f(x,y)$ = probability that y interceptors will break up a formation of x aircraft

whether the break-up is in the sense of model A or of model B. Obviously the function $f(x,y)$ is not necessarily the same for the two models.

Model A probably would apply only to cases involving low-speed, low-performance aircraft, while model B is preferable for high-speed engagements. Of course many other models, including combinations of these, are conceivable.

In both models the measure of effectiveness of the raid will be the expected number of penetrating aircraft, that is, aircraft not killed (in the sense of (i) of section 1) by interceptors or guided missiles. If in one engagement x enemy aircraft are met by y interceptors, the expected number $E(x,y)$ of penetrating aircraft is

(1) $E(x,y) = x - yp_1 - f(x,y)(x-yp_1)p_2 - [1 - f(x,y)]mp_2$

for model A, or

(2) $E(x,y) = x - yp_1 - f(x,y)mp_4 - [1 - f(x,y)]mp_3$

for model B.

These formulas are exact only if the smallest possible number of planes penetrating the interceptor defense (i.e. $x - y$) is $\geq m$, that is, if there must be at least m planes remaining for the guided missiles; they may, however, serve as an approximation if $x - yp_1 > m$, that is, if it is "probable" that at least m planes will remain. We therefore assume that $x > m + yp_1$ in every engagement that we consider, or simply that $\frac{M}{2} > m + Np_1$. This condition defines the word "large" used in the beginning of the introduction. (It should be pointed out that the assumption in model A that a guided missile salvo can be fired against every aircraft of a broken formation may be unrealistic in the case of very large raids; in this case, however, strategy is not likely to have much influence on the outcome.

5. Solution of the Problems Posed by Models A and B.

The problem can now be set up in the language of the theory of games. The strategies for player I (attacker) are:

1. Send all M aircraft in one raid
2. Send two raids of $M/2$ aircraft each

Strategies for player II (defender) are:

1. Send all N CAP against the first raid
2. Send $N/2$ CAP against the first raid, holding the remainder for a possible second raid.

The payoff matrix is then the following:

Strategies of Player II

		1	2
Strategies of Player I	1	$E_{11} = E(M,N)$	$E_{12} = E(M,N/2)$
	2	$E_{21} = E(M/2,N) + E(M/2,0)$	$E_{22} = 2E(M/2,N/2)$

Here the values of $E(M,N)$, etc., are computed according to formula (1) or (2) depending on the model used.

In solving the game we assume, as noted earlier, that $\frac{M}{2} - Np_1 > m$, and also that

$$f(M,N) \geq f(M,N/2)$$

$$f(M/2,N) \geq f(M/2,N/2) \geq f(M/2,0)$$

$$f(M/2,N/2) \geq f(M,N/2).$$

For model B we assume in addition that $p_4 \geq p_3$. It then follows that $E_{12} \geq E_{11}$ and $E_{12} \geq E_{22}$, so that a solution can be written as follows for both models:

Case	Optimal Strategy Player I	Optimal Strategy Player II
$E_{22} \geq E_{21}$, $E_{11} \geq E_{21}$	1	1
$E_{22} \geq E_{21}$, $E_{11} < E_{21}$	2	1
$E_{22} < E_{21}$, $E_{11} \geq E_{21}$	1	1
$E_{22} < E_{21}$, $E_{11} < E_{21}$	$(X,1-X)$	$(Y,1-Y)$

Here $(X,1-X)$ indicated a mixed strategy, using strategy 1 with probability X and strategy 2 with probability $1-X$. We have

$$X = (E_{21} - E_{22})/(E_{21} - E_{22} + E_{12} - E_{11})$$

$$Y = (E_{12} - E_{22})/(E_{21} - E_{22} + E_{12} - E_{11})$$

(In certain cases where equality signs hold, the solution is not unique.)

The first two cases in the table of strategies seem unlikely to occur; the requirement that $E_{22} \geq E_{21}$ means that if the enemy splits his forces the defense may nevertheless do better to send all of his CAP against the first raid. This could conceivably be the case if N interceptors had a very much higher probability of breaking up a formation of M/2 aircraft than would N/2 interceptors. The third case in the table seems most likely to occur in practice. The following examples provide solutions for a case using what seem to be reasonable values of the parameters. These examples, incidentally, illustrate the fact that solutions in model A depend on M, while those in model B do not except through the function $f(x,y)$, which is probably not sensitive to M when M is large.

Example for model A. Let us suppose that $N = 20$, $m = 10$, $p_1 = 0.3$, $f(M,N) = 0.3$, $f(M,N/2) = 0.2$, $f(M/2,N) = 0.4$, $f(M/2,N/2) = 0.3$, and $f(M/2,0) = 0.1$. [Note. The last number is not 0, as the formation may break up accidentally.] We must assume that $M/2 > m + Np_1$, which means that $M > 32$. If we feel that the enemy will not send more than 32 aircraft, the decision as to disposal of the 20 CAP is not so difficult; at any rate it must be made by another method. We find at once that $E_{22} \geq E_{21}$ only if $M \leq 8$, so we must be in the third or fourth case of the table of strategies. We find that

$$E_{11} \geq E_{21} \text{ if and only if } M \leq 148.$$

Hence if we believe that $M \leq 148$ we send all CAP against the first raid. Otherwise we use a mixed strategy, sending all CAP against the first raid with probability

$$Y = (10.8 + 0.1M)/(3.4 + 0.15M).$$

That we do not have to know M with great accuracy is shown by the following table of variation of Y with M:

M	32 to 148	200	300	500
Y	1	.92	.85	.78

Example for Model B. Using the same numbers as in the preceding example, together with $p_3 = 0.1$ and $p_4 = 0.5$, we find that $E_{22} < E_{21}$ and $E_{11} > E_{21}$, so that the enemy should send a maximum raid and all CAP should be sent against it.

6. The Case of Unequal Threats.

The methods of the preceding section can be extended to the situation wherein the enemy has two groups of aircraft of unequal threat which can be brought in singly or simultaneously. This could occur because of different weapons delivered by the two groups or because of a single available atomic bomb. The resulting game has been solved, using the point of view of model B, but the solution requires that the defense know the relative threats of the two groups of aircraft (although not which one is first in case they are split.) A considerable number of cases must be enumerated, and the solution will not be considered here.

7. A Model in which Formation Break-up is a Strategy.

We consider now a model in which the breaking up of a formation is not a matter of probability but depends only on the attacker's doctrine regarding maneuvering. That is, we assume that the enemy can decide to maneuver or not to maneuver while being attacked by interceptors. If he maneuvers, he presents a more difficult target to the interceptors, but his formation will probably not remain tight enough to prevent resolution by the guided missile radars. The important parameters here are p_1, p_2, p_3, p_4, defined as follows:

	Enemy maneuvers during interception phase	
	Yes	No
kill probability for each interceptor	p_1	p_2
kill probability for each guided missile salvo	p_3	p_4

We shall assume that $p_1 < p_2$ and that $p_3 > p_4$. If either of these is false and the other true, there is no problem.

Let us suppose that an enemy raid of x aircraft is met by y interceptors. Those enemy aircraft which survive the interceptor phase are attacked by a fixed number m of guided missile salvos. If we assume that x is large enough so that $x - yp_2 > m$, the expected number of aircraft penetrating both defenses is, at least approximately,

(1) $x - yp_1 - mp_3$ if enemy maneuvers

(2) or $x - yp_2 - mp_4$ if enemy does not maneuver.

If the enemy had all information, therefore, he would maneuver if and only if

$$x - yp_1 - mp_3 > x - yp_2 - mp_4,$$

that is, he should maneuver if and only if he is met by more than y_1 interceptors, where $y_1 = m(p_3 - p_4)/(p_2 - p_1)$. The fact that y_1 does not depend on x is due to the assumption that x is large compared to y and m.

In practice, the enemy probably cannot determine, in time to make a decision, the exact number of interceptors. His radar resolution may be assumed to be such that he can distinguish only among "one", "few", and "many" interceptors. To express this fact in a simple arithmetical way we postulate a number $y_0 \leq y_1$ such that

when $y \leq y_0$ the enemy knows that $y \leq y_1$, but

when $y > y_0$ he cannot tell.

We shall assume that the enemy doctrine is to instruct each raid before going in either to maneuver (if and only if met by more than y_0 interceptors) or not to maneuver in any case.

We suppose as before that there are N airborne CAP and that the enemy has a total force of M aircraft which are brought in either all at once or in two equal raids so close together in time that detection of the first raid does not occur early enough to allow additional interceptors to become airborne and intercept the second raid. Previously it has been assumed that m guided missile salvos could be fired against each raid, but here we take a more general case, assuming that only km (where $0 \leq k \leq 1$) salvos can be fired against the second raid.

Strategies: We allow the defense $N+1$ strategies, numbered 0, 1, 2, ..., N according to the number of CAP sent against the first raid. The enemy is given six strategies, numbered as follows:

Send all aircraft in one raid with instructions

1. to maneuver
2. not to maneuver

Send aircraft in two equal raids and instruct

3. both to maneuver

4. the first to maneuver, the second not to maneuver
5. the first not to maneuver, the second to maneuver
6. both not to maneuver.

We let G_{ij} be the payoff, in expected number of penetrating aircraft, when the enemy uses strategy i ($i = 1, 2, \ldots, 6$) and the defense uses strategy j = ($j = 0, 1, 2, \ldots, N$). The values of the G_{ij} are given below. To show how they are computed let us use G_{4j} as an example. Here the enemy sends $M/2$ aircraft in raid 1 with instructions to maneuver if met by more than y_o interceptors, and $M/2$ aircraft in raid 2 with instructions not to maneuver. If the defense uses a strategy $j \leq y_o$, neither raid maneuvers and the value of G_{4j}, computed from (2), is $M - Np_2 - mp_4 - kmp_4$. If $j > y_o$, the first raid maneuvers and G_{4j}, computed from (1) and (2), is

$$(M/2 - jp_1 - mp_3) + [M/2 - (N - j)p_2 - kmp_4].$$

Proceeding in similar fashion we obtain the following:

$$
\begin{aligned}
G_{1j} &= M - jp_2 - mp_4 && \text{if } j \leq y_o \\
&= M - jp_1 - mp_3 && \text{if } j > y_o \\
G_{2j} &= M - jp_2 - mp_4 && \text{for all } j \\
G_{3j} &= M - jp_2 - mp_4 - (N-j)p_1 - kmp_3 && \text{if } j \leq y_o \\
&= M - Np_1 - (1+k)mp_3 && \text{if } y_o < j < N-y_o \\
&= M - jp_1 - mp_3 - (N-j)p_2 - kmp_4 && \text{if } j \geq N - y_o \\
G_{4j} &= M - Np_2 - (1+k)mp_4 && \text{if } j \leq y_o \\
&= M - jp_1 - mp_3 - (N-j)p_2 - kmp_4 && \text{if } j > y_o \\
G_{5j} &= M - jp_2 - mp_4 - (N-j)p_1 - kmp_3 && \text{if } j < N - y_o \\
&= M - Np_2 - (1+k)mp_4 && \text{if } j \geq N - y_o \\
G_{6j} &= M - Np_2 - (1+k)mp_4 && \text{for all } j.
\end{aligned}
$$

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It is easy to show that

$$G_{1j} \geq G_{3j} \quad \text{and} \quad G_{1j} \geq G_{4j} \quad \text{for all } j$$

$$\text{and} \quad G_{2j} \geq G_{5j} \quad \text{and} \quad G_{2j} \geq G_{6j} \quad \text{for all } j,$$

the equality signs here being impossible unless, among other things, $k = 0$. It follows that <u>in this model the enemy should use the maximum raid, regardless of the defender's strategy or the values of the parameters.</u> It is evident, then, that the defense should use strategy N, and this can be shown from the formulas for G_{1j} and G_{2j}. Thus the defense should use all CAP against the first raid, and the enemy should maneuver if and only if he thinks $N > y_1$.

8. <u>The Case of One Large Raid.</u>

From this point we assume that the enemy delivers all his aircraft (about 200 to 300) in one raid, but that he is not able to make all these aircraft arrive simultaneously. The essential concept to be used is that of a "unit" of aircraft. We shall suppose that enemy aircraft arrive in units of about 6 or 8 aircraft each; the actual number here is not important, except that in this treatment we must suppose that the number of units, say 30 - 50, is large enough to be treated statistically. (Unless information to the contrary is available, one may as well suppose that these units are of the same composition. However, if the defender expects, say, one collection of units of dive bombers and one collection of units of high altitude bombers, he can decide how he wishes to divide his forces between the two attacks and then consider each attack as a separate problem of the type below.)

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As a guiding principle for the defender, we shall suppose that the best disposition of his interceptors is that which presents, as nearly as possible, an equal threat to each unit of enemy aircraft. The defender's principal problem, then, is to decide how many interceptors to vector against each enemy unit, that is to decide on a unit size of his own.

9. Notation.

We first define a region A around the task group determined by the defender as follows: as long as a given enemy unit has not penetrated A it is not too late to vector airborne interceptors toward it, but once the unit has penetrated A it is considered too late to do so. (Here, as well as below, it will be convenient to use "airborne" to mean at altitude, over or near the task group.) Any enemy units which are not detected until already in A are irrelevant to the problem under consideration, and so all enemy units mentioned below will be assumed to have been detected before reaching A.

We let t_o stand for the time of first detection. As before, N will be the number of airborne CAP at time t_o, but M will be the number of enemy units (not aircraft). The size of these units, assumed constant, is of course important to the outcome of the whole engagement, but does not affect the decisions considered here.

Two important functions are shown (qualitatively) in Figure 1.

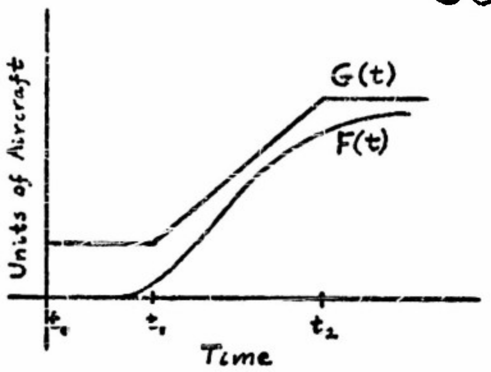

Figure 1

The curve $F(t)$ shows the number of enemy units that will have penetrated A by time t unless prevented from doing so by interceptors. If $F(t)$ is more nearly vertical the enemy is more nearly achieving his objective of simultaneous arrival. The shape and location of the $F(t)$ curve will depend on M, on the distance from A (at detection time) of the first unit to be detected, and on various factors which determine the enemy's ability to synchronize his units, such as the number and location of his airbases and the state of training of his personnel.

The function $G(t)$ represents the number of units of airborne interceptors that the defense has put up by time t. At time t_0 steps are immediately taken to launch additional interceptors, and, starting at a time t_1, there are b additional interceptors airborne per minute. The graph of $G(t)$ consists of three parts: phase 1, time t_0 to t_1, a horizontal line representing the number of airborne units at t_0, phase 2, time t_1 to t_2, a line of slope b representing units of interceptors becoming airborne after t_1; and phase 3, another horizontal line, beginning at time t_2 when the defense has either run out of interceptors or has decided not to use any more in the present engagement.

It may occur that the CAP aircraft, designed for endurance, are less effective as fighters than the interceptors launched later. It will be assumed below that this is not the case, or rather, that if the CAP aircraft are, say, only three-fourths as effective as the others, then each four CAP aircraft are treated as though they were only three interceptors.

10. The Method.

The defender's chief problem will be to estimate the curve $F(t)$. Once a number of such large engagements have taken place, operational data should be available which will provide information about $F(t)$. With this information, combined with knowledge of number and location of enemy airbases in the vicinity, a fairly good estimate of $F(t)$ should be possible. In the absence of operational data, the defender must be content with some assumed form; he may assume, for example, that arrival times of enemy units are normally distributed about their time of estimated arrival, with a standard deviation that depends on the distance from the enemy bases to the task group.

It will be assumed that the basis for decision-making will, in general, be the principle that the curve $G(t)$ must always remain above the curve $F(t)$ when the two are plotted together; otherwise, of course, some enemy units, though detected before entering A, will not be intercepted. Possible exceptions to this rule will be noted as we proceed.

Given certain forces at the onset, that is, given a certain amount of CAP and the ability to put aircraft into the air at a certain maximum rate, the defender has no way of increasing his capabilities as measured by total number of interceptors airborne by a given time. He can, however, control the graph of G(t) somewhat by changing his unit size. If his unit size is too large, the G(t) graph will not stay above the F(t) graph; on the other hand, the unit size should be as large as possible in terms of the defender's capabilities. (Given certain kinds of information, it may even be desirable to let the G(t) graph drop below the F(t) graph at times, provided the increased effectiveness due to a greater unit size compensates for the damage done by unintercepted enemy units.)

Let us consider a hypothetical example with numbers chosen simply for arithmetical convenience. The defender has 36 airborne CAP at time $t_0 = 0$, and at $t_1 = 10$ minutes he can start putting up 12 additional interceptors per minute and continue this for 10 minutes.

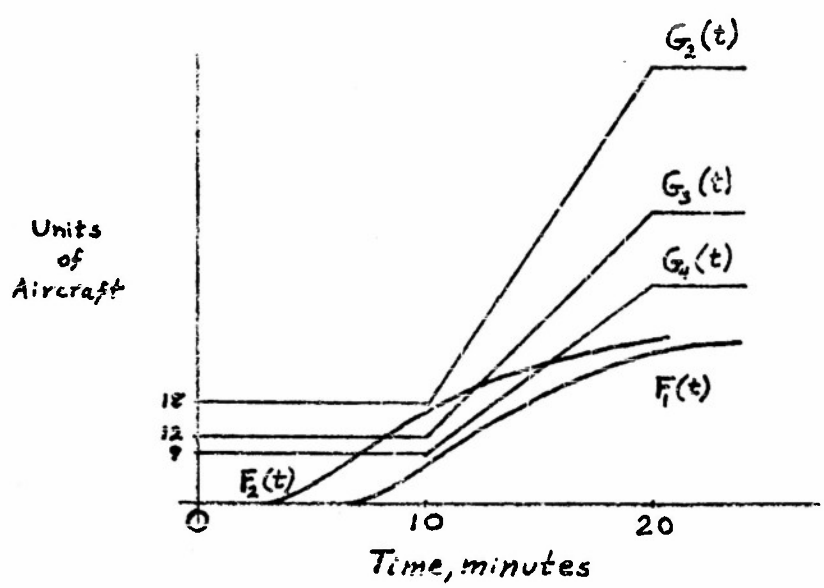

Figure 2

Figure 2 shows how he can vary his $G(t)$ graph by varying his unit size, $G_i(t)$, for $i = 2, 3, 4$, representing the use of i interceptors per unit. Two possible enemy curves, $F_1(t)$ and $F_2(t)$, are also shown. If the distribution of enemy units is $F_1(t)$, the defender can use $G_4(t)$, that is, he can send his interceptors in units of 4 against each enemy unit. If the distribution of enemy units is $F_2(t)$, the defender can safely send only two interceptors against each enemy unit during phase 1, but he can increase this figure to 4 during phase 2.

There has been evidence in the past that the effectiveness of an interception increases with the distance of the interception from the task group. If this continues to be the case, the lowest $G_i(t)$ curve that lies wholly above the $F(t)$ curve will not necessarily be the best choice. However, until operational data are available for the high-speed engagements of the future, there is no way to weigh the factors involved here, so that one may as well assume what may turn out to be the case, namely that there is little or no advantage to very early interception.

In introducing the method it has been supposed that the defender knows the graph of $F(t)$ exactly. In practice this is of course impossible and we next consider the problem of estimating $F(t)$.

11. The First Estimate.

In the absence of any data to the contrary, we shall assume that enemy units come in at times which are normally distributed about an unknown mean time μ with standard deviation σ. We shall suppose that detection probability follows a "definite range law"; that is, we eliminate the problem of detection probability by assuming that detection always occurs at a fixed range. This is equivalent to the assumption that detection times, rather than arrival times, are normally distributed. In practice it is probably true that neither time is normally distributed, and in any case we are merely trying to obtain a first rough estimate.

It should be pointed out first of all that, if the defender knows μ (which he never will) and if he knows σ (which he may be able to estimate), then even in these happy circumstances he must still cope with the particular $F(t)$ curve (step-function) of the present engagement. This curve represents not a population, but a sample from the population, and its mean and standard deviation will almost surely not be those of the population. All this underlines the fact that <u>it is not possible to choose the "i" in $G_i(t)$ so as to be absolutely certain of keeping the $G(t)$ graph above the $F(t)$ graph.</u>

In practice, the defense will have perhaps a fairly good estimate of σ, only a rough estimate of M, and no idea at all of μ. Let T_r be the time at which the expected proportion of the population that has been detected is r, $0 \leq r \leq 1$. Suppose the first enemy unit is detected at time t_0. A decision must be made almost immediately. About the only kind of question that can be answered at this moment, however, is of this type: "What is the earliest time that we must expect $\mu = T_{1/2}$ (or T_r for some other r), provided we are willing to be in error in not more than, say, 5 percent of all cases?" To answer this we look for a time T such that

$$Pr(T_r - t_0 < T) = .05$$

The probability that $T_r - t_0 < T$ is simply the probability that t_0 (and hence every detection time in the sample) is greater than $T_r - T$. Hence

$$Pr(T_r - t_0 < T) = [Pr(t > T_r - T)]^M$$

where t is the detection time of a randomly chosen unit and M is the total number of units. To find T we must solve

$$Pr(t > T_r - T) = (.05)^{1/M}$$

Using a table of the normal curve, we find that

$$\begin{aligned} T &= T_r - \mu + 1.31\,\sigma \quad \text{if} \quad M = 30 \\ &= T_r - \mu + 1.46\,\sigma \quad \text{if} \quad M = 40 \\ &= T_r - \mu + 1.57\,\sigma \quad \text{if} \quad M = 50 \end{aligned}$$

The value of $T_r - \mu$, for any r, can be found in terms of σ from a table of the normal curve.

As an example, let us suppose that the first unit is detected at time 0 and is expected to enter A in 4 minutes. If $\sigma = 5$ minutes, then $T_r - \mu = 6.55$, 7.30, or 7.85 according as $M = 30$, 40, or 50. This means that the planned arrival time μ is, and we can say this with "95% certainty", not earlier than 10.5, 11.3, or 11.8 minutes if $M = 30$, 40, or 50, respectively. We have not said that the probability is 0.95 that the first half of the units will not be in earlier than these times, but it is clear that the defense will be well prepared if he expects about half the units to be in A by time 11 or 12 minutes.

12. Revised Estimates.

Once the raid is well under way, a cumulative record of enemy units detected should show fairly well whether early decisions should be revised. On a sheet of graph paper the $G_i(t)$ curves can be drawn beforehand:

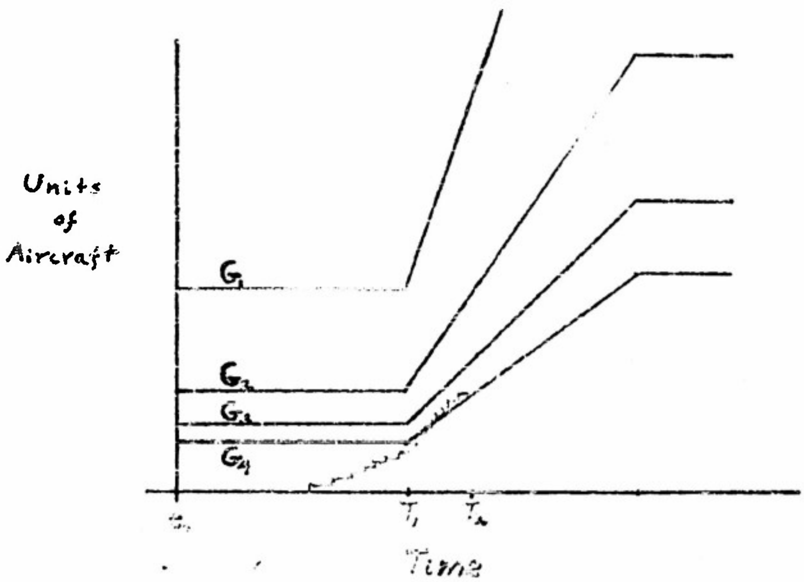

Figure 3

As each enemy unit is detected during the raid, its time of arrival in region A can be estimated and the unit is then added as another step in the graph (Figure 3). Suppose we are at time T_1 and have been using $G_4(t)$, that is, sending out CAP in units of 4. The step graph indicates times of arrival in A of units already detected and so the graph extends beyond the present to a point T_2. Evidently it will no longer be possible to send aircraft in units of 4, or even 3. In this case 2 will apparently be a safe unit size, but it must be remembered that that part of the step graph between T_1 and T_2 is still subject to change, since the nearest of several approaching units is not necessarily the first one to be detected. In fact, as we move along the step graph into the future, it is to be expected that an increasingly small fraction of units that will ultimately lie there have already been detected. Hence there is likely to be an apparent leveling off of the graph near the time T_2, and it would be dangerous for the defender to take this at face value. On the other hand, if he makes due allowance for this effect, he should obtain useful information from the cumulative graph.

13. Statistical Estimation.

The preceding section showed how the defense can gather and display information during the raid that will help him in deciding whether to revise his unit size. We now investigate the possibility of using methods of statistical estimation to help in this decision. Some theory in this connection may be found in references [1], [3], and [4], the tables used being in reference [2].

It is brought out below that one cannot hope to estimate all three of the parameters M, μ, and σ from the detection times of the first few detected units. However, if there is previous knowledge of at least one of these parameters, it may be possible to make useful predictions during the engagement.

The first few detections made during a raid constitute a "truncated sample" and from this it is possible to estimate the mean and standard deviation of the population. One would like to know at once, of course, how good these estimates are. The above references give asymptotic variances of the estimates, but these are useful only for large samples and there is no assurance that the samples obtained in the present problem will be large enough for the asymptotic variances to provide useful approximations. The raids considered here consist of something like 40 to 50 enemy units, and there is little point in estimating unless a sample of 10 or 20 provides useful estimates.

In order to provide a preliminary estimate of the precision of the estimating procedure, twenty samples of forty numbers each were drawn from a table of random normal deviates. These numbers have mean 0 and standard deviation 1, so that the forty numbers in a given sample represent arrival times of enemy units attempting to arrive at time 0 and doing so with errors of standard deviation 1.

It was first attempted to estimate $\mu(=0)$ and $\sigma(=1)$ and $M(=40)$ from the first 10 and also from the first 20 observations in each sample. These estimates were so poor as to be useless, even in the case of 20 observations, where 6 of the 20 estimates of M were over 80 and ranged to over 800.

It is intuitively evident, however, that if σ is known the estimates of μ and M should be greatly improved. Table I shows the results, for twenty samples, of estimating μ and M, given σ. Columns 2 and 3 give estimates of μ and M as obtained from the first ten of the forty arrival times in each sample; entries such as ">3.0" mean "greater than 3.0" and indicate that some number involved was beyond the range of the table. Columns 4 and 5 give similar estimates as obtained from the first 20 observations in each sample. Columns 6 and 7 show the effect of an error in the estimate of σ by showing estimates of μ and M, for 20 observations, found on the incorrect assumption that $\sigma = 0.75$.

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Table I

Estimates of $\mu(=0)$ and $M(=40)$, given σ

Sample Number	10 Observations $\sigma = 1$		20 Observations $\sigma = 1$		20 Observations σ misestimated as 0.75	
	μ	M	μ	M	μ	M
1	-0.3	42	-0.5	29	-0.8	23
2	0.9	250	-0.1	48	-0.6	30
3	0.6	71	0.6	74	-0.4	29
4	-0.3	24	0.0	35	-0.4	24
5	0.4	125	-0.2	40	-0.7	26
6	-0.4	24	-0.1	32	-0.6	23
7	>3.0	>430	-0.1	37	-0.6	25
8	-0.1	37	0.0	42	-0.6	26
9	-0.9	250	0.0	42	-0.6	26
10	-0.8	16	0.1	48	-0.5	29
11	>1.3	>430	0.4	48	-0.2	29
12	0.0	48	-0.2	35	-0.3	30
13	0.1	48	0.1	40	-0.5	25
14	0.1	56	0.1	59	-0.5	35
15	1.0	83	0.9	83	0.0	35
16	0.3	42	0.5	64	-0.1	37
17	1.0	280	0.3	59	-0.3	32
18	>0.8	>430	-0.1	53	-0.6	32
19	0.8	83	0.4	48	-0.1	30
20	0.3	42	0.6	83	-0.1	40

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In the case of 10 observations it appears that estimates of μ are usually fair, although estimates of M are quite poor. With 20 observations, estimates of μ are good, while those of M are fair. In general these estimates are good enough so that further sampling is warranted provided the general model is considered useful.

The effect of an error in the original estimate of σ is particularly important. If enemy units actually arrive in the manner considered here, it should be possible to put a lower bound on σ, the enemy being capable of increasing σ either by accident or design, but unable to diminish it. Underestimation of σ is therefore of more interest than overestimation, as the defender can take σ to have its smallest likely value and know that he is underestimating. The last two columns of table I show the effect of underestimating σ by 25%. The resulting errors in the estimates of μ are not bad, but the estimates of M are consistently too small and so can be dangerous. In general, if σ is taken to be the smallest value it could conceivably have, the attacker can only increase σ, and this has the effect, so far as μ is concerned, of telling the defense to be ready for the main part of the attack a little sooner than it will actually come. On the other hand, the defense could be easily fooled (if the enemy, say, should increase σ purposely) into thinking the raid is smaller than it is if the defense bases estimates of M strictly on this method.

Finally there is the possibility that the defense has, from intelligence, an estimate of the enemy potential in terms of number of aircraft. After ten or twenty units have been detected, some of the earlier ones may have drawn near enough so that the unit size can be estimated; from this and the estimate of total aircraft one may estimate M.

If M were known exactly, there would not be much left for the defense to want to know, and the principal point of interest here, therefore, is the effect of misestimating M. In Table II this effect is studied for the same samples used before. Here h stands for the proportion of units not detected by the time estimates are made; since M = 40, the true value of h is therefore 0.75 or 0.5 according as we are estimating from 10 or from 20 observations. For simplicity, incorrect values of h were taken to be 0.7 and 0.8 for the case of 10 observations, and 0.4 and 0.6 for the case of 20 observations. Since one is particularly concerned here with large errors, Table II summarizes the results by showing the greatest error (and, in parentheses, the next greatest) occurring over the twenty samples in each case.

Table II

Largest and Next Largest Errors in Estimates of μ and σ from 20 Samples, Assuming Various Values of M.

Estimate of M	10 Observations		20 Observations	
	μ	σ	μ	σ
40 (correct)	.83 (.64)	.52 (.49)	.42 (.34)	.27 (.25)
50	.93 (.73)	.56 (.53)	.61 (.53)	.34 (.28)
33	.78 (.75)	.51 (.50)	.56 (.48)	.29 (.24)

All of these errors are less than 1. In the case of 20 observations, the worst error obtained is for μ, an error of

0.61. If σ were 5 minutes, this would mean that the greatest error (in twenty samples) in estimating μ was 3 minutes. It appears that incorrect estimation of M, when mistakes are of the order of those considered here, does not do much harm to the estimates of μ and σ.

14. Force Requirements.

The concern of this paper has been with decisions that are made after a raid is detected. Any decisions made in the planning stages, however, such as the decision on the number of CAP to keep airborne, are obviously aided by knowledge of steps that must be taken during an engagement.

The graphical treatment involving the F(t) and G(t) curves can be helpful in the planning stages. At this time the defender has a great deal more control over the $G_1(t)$ graphs than he does after an engagement has begun. Figure 4 shows, for example, two possible forms of $G_1(t)$, i.e., different forms that can be produced without changing the unit size. Here

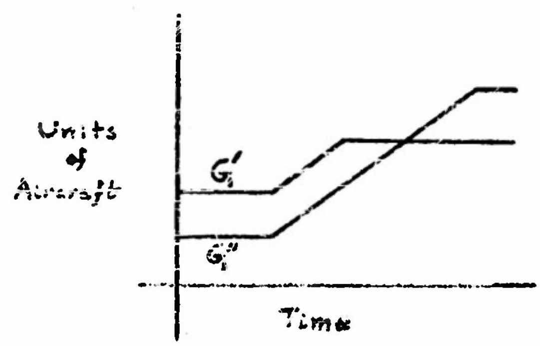

Figure 4

$G_1''(t)$ represents the case wherein only half as much CAP is kept airborne as in $G_1'(t)$. Thus $G_1''(t)$ is weaker during the first phase of the attack but later becomes stronger by virtue of the fact that a task group that carries less CAP can carry more other fighters and so can keep putting them into the air longer after an engagement begins. Given the space restrictions, an upper limit to the number of enemy units, the desired size for defense units, and some knowledge of the detection capabilities of the defense, one can approximate the answer to the question, "How many CAP aircraft should be kept airborne when raids are possible?"

15. Conclusion.

One of the uses of a mathematical analysis such as the present one is to point out areas of knowledge, or simply parameters, where useful operational information is needed. Two such areas that are emphasized in this paper may be described by asking the following two questions:

1. How do interceptors "break up" a formation of attacking bombers, and what are the results of doing so?
2. When a large number of aircraft attempt simultaneous arrival at a given location, in what way do they actually arrive?

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